On the control of mechanical systems with dynamic backlash

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Abstract

The focus of this work is the analysis and control of a simple rigid body mechanical system with dynamic backlash. Contrary to most of the existing work in the control literature we explicitly treat all the dynamic and non-linear characteristics of this system. A desired symmetric periodic orbit locally stable is generated by a PD control scheme. In order to enlarge the basin of attraction of this orbit we propose the use of a hybrid control in addition to the PD control. This work finds potential application in several areas including the control of kinematic chains with joint clearance and vibro-impact systems.

1 Introduction

Backlash is one of the most important non-linearities that taxes the control strategies implemented in the industrial machines and degrades the overall performance of the machines. It causes delays, oscillations, and consequently gives rise to inaccuracies in the position and velocity of the machine. In extreme cases, backlash related effects can help set in an extremally complicated system behavior thereby making it completely intractable from the point of view of the controller.

Backlash commonly occurs in bearings, gears and impact dampers. It arises from unavoidable manufacturing tolerances or are often deliberately incorporated in the system to accommodate thermal expansion [2]. Fig 1 presents the sketch of a simple system with backlash which uses a rigid body contact/impact model.

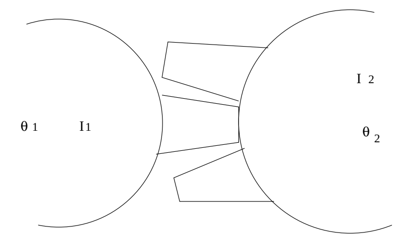

Figure 1: System with backlash

The control of system with backlash has been investigated by several authors and there exists different approaches. [10] analyzed the problem and proposed to model backlash as a hysteresis function between the output and input positions of the system. This is a geometric model in which the system input is θ_1 and the output is θ_2. Given that the dynamic effects of the collisions are not taken into account in this model, it cannot describe the real dynamics of the system in Fig. 1. This backlash model does not approximate the real physical system in Fig. 1 as it makes the un-physical assumptions that the shocks are purely inelastic and that the ratio of the inertias of the two interacting masses, $\frac{I_1}{I_2}$, is zero. They propose an algorithm for the compensation of the backlash dynamic that use an adaptive control strategy with a high gain.

Other approach is used by [8] which formulated the dy-

*Supported by CONACYT-México

namic equations of motion for an impact pair including compliance at the contact. In his model, τ is the input and θ_2 the output. It is assumed that $I_1 = 0$, i.e. the system is an inertia free elastic shaft system with backlash. Contrarily to [10] their linear control input uses a low gain when the system evolves inside the clearance.

In [1] the authors investigate the control of a system with dynamic backlash considering the collisions. With respect to Fig. 1 their control model considers the torque τ_1 as the input and the position θ_2 as the output. In their control model, the effect of impact of the gears on the dynamics of the first gear (in Fig. 1) is considered as a disturbance. In order to explore the limit cycle behavior of the system they base their study on the describing function techniques.

Several other backlash models have been proposed and studied in the mechanical engineering literature and, in particular, in relationship with the so-called impact damper. [2, 9] studied the dynamic response of simplified rigid-body impacting systems. They showed the existence of complex dynamics including different types of periodic trajectories, bifurcations, and chaotic motion.

In this work the impact damper (see Fig. 2) is used as a simplified model of backlash for feedback control purposes incorporating the dynamical effects of impacts. We use a rigid body model which is justified by typical numerical values of 10^{10}N/m for the contact/impact stiffness that has been reported in the literature [5, 4].

We wish to underline here that in contrast to the models of [10] and [8] we consider the torque τ_1 as the input and the angular position θ_2 as the output. In the linear backlash model as shown in the Fig. 2 this corresponds to U and x_1, respectively. In our opinion this is the most practical model for control purposes in a machine with clearance. Our study finds potential application in the control of robot manipulators whose performance may be degraded because of the presence of clearance in the joints [5].

In this work we apply PD control strategy to this system and identify the realizable stable symmetric periodic solutions. This linear control law is active only around a periodic solution. In order to arrive at this solution from given initial conditions (which often we have no control upon) we apply a different control scheme. The underlying philosophy of this strategy is similar to the one proposed in [6] where the second control law is used to guide the trajectories to the basin of attraction of the linear control. We use a control consisting of a constant input and appropriately timed impulses. We show that with a minimum number of two impulsive inputs the primary mass can be brought from any initial condition to a stable periodic symmetric orbit, which is then preserved with the PD control law.

2 Controlled impact-damper model

A schematic diagram of the mechanical system under consideration is shown in Fig. 2, which consists of a secondary mass m_2 subject to an external control input U and a primary mass, m_1, which is constrained to move in a slot inside the mass m_2. The supposed frictionless motion of m_1 is instigated by collisions with m_2 which occur intermittently because of the clearance $2L$. The play comes from the size of the slot in m_2 being larger than the size of m_1. Due to the absence of friction, the velocity of m_1 remains constant between two consecutive impacts. The physical contacts may repeat many times, leading to a finite or infinite number of collisions.

This idealized model is called an *impact pair*. It is a simplified version of many typical mechanical systems with clearances. Because of its simplicity, it has been used frequently as a basic model for the study of mechanical systems with clearances [2, 9, 4] and references therein. Although it is an approximate model, it exhibits the typical behavior found in such systems and has an extremely rich dynamics.

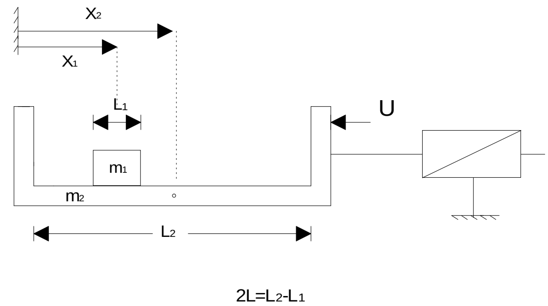

Figure 2: Physical model

The equations governing the motion between two successive impacts are:

$$\begin{array}{l} m_1\ddot{x}_1 = 0 \\ m_2\ddot{x}_2 = U \end{array} \qquad \text{for } |x_1 - x_2| < L \tag{1}$$

Notice that this is an under-actuated system in the sense that there is only one control input and two degrees of freedom. Also this system belongs to the class of systems called hybrid dynamical systems. The motion of the system at the instant of impact must satisfy the linear momentum conservation equation,

$$m_1(\dot{x}_1(t_k^+) - \dot{x}_1(t_k^-)) + m_2(\dot{x}_2(t_k^+) - \dot{x}_2(t_k^-)) = 0 \quad (2)$$

where t_k^+ and t_k^- denote the velocities just after and just before the k^{th} impact. All impacts are supposed, reasonably for metals, to be instantaneous and described by a coefficient of restitution, e, representing the energy lost during collision [3]. From the definition of the coefficient of restitution, which gives the ratio of the relative velocity after and before the impact, one has

$$\dot{x}_1(t_k^+) - \dot{x}_2(t_k^+) = -e(\dot{x}_1(t_k^-) - \dot{x}_2(t_k^-)) \quad (3)$$

From the equations (2) and (3) and considering that the positions of the masses are not changed during the impact, the following relations can be obtained,

$$\begin{pmatrix} x_1(t_k^+) \\ x_2(t_k^+) \\ \dot{x}_1(t_k^+) \\ \dot{x}_2(t_k^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\mu-e}{1+\mu} & \frac{(1+e)}{1+\mu} \\ 0 & 0 & \frac{(1+e)\mu}{1+\mu} & \frac{1-\mu e}{1+\mu} \end{pmatrix} \begin{pmatrix} x_1(t_k^-) \\ x_2(t_k^-) \\ \dot{x}_1(t_k^-) \\ \dot{x}_2(t_k^-) \end{pmatrix} \quad (4)$$

where $\mu = \frac{m_1}{m_2}$ is called the mass ratio.

Let us consider the case where $L = 0$, i.e. the clearance is zero and $x_1 = x_2$ (there are no impacts). Then this second order system is easily controlled via a PD controller of the form

$$U = m_2\ddot{x}_{2d} - k_v\dot{\tilde{x}}_2 - k_p\tilde{x}_2, \qquad k_p > 0, k_v > 0 \quad (5)$$

with $\tilde{x}_2 = x_2 - x_{2d}$ and x_{2d} is a desired trajectory. This control is applied only on m_2.

In the following we shall choose x_{2d} as a cosine function,

$$x_{2d} = x_m \cos \omega t \quad (6)$$

so that the closed-loop behavior of the controlled system between impacts is given by,

$$\begin{aligned} m_1\ddot{x}_1 &= 0 \\ m_2\ddot{x}_2 &= -k_v\dot{x}_2 - k_p x_2 + \beta\cos(\omega t + \zeta) \end{aligned} \quad (7)$$

For the system with $L \neq 0$, this control is effective only between impacts. Since the impacts are considered instantaneous, this control has no effect during the impacts. However the impacts influence the evolution of m_2 and m_1. It is important to observe that the motion of m_1 can only be indirectly controlled via impacts.

We build the periodic trajectories with this control law based in the observation that the equations are linear between collisions; thus explicit solutions can be obtained. These solutions are concatenated at the instant of the impact with an impact rule to obtain a complete solution. With this design philosophy we find the necessary conditions for obtaining periodic symmetric orbits with two collisions per period (this is the simplest solution possible). We show that is possible select the appropriates values of the gains so that the control law creates a desired stable trajectory. These trajectories correspond to fixed points of the Poincaré map. The stability analysis is done computing the Jacobian matrix of the Poincaré map at the periodic orbits and finding the eigenvalues. Those periodic trajectories are locally stable with an a priori unknown basin of attraction. The details of calculations are in [7]. It is necessary to design a complementary control strategy to enlarge the basin of attraction of the whole scheme. This is the subject of the next section.

3 Hybrid control

This section is devoted to the study of a control strategy which allows us to bring the trajectories into the basin of attraction of a desired periodic motion $p_d(t)$. It is noticeable, that due to the non smooth behavior of the system, $p_d(t)$ cannot be chosen as $x_{2d}(t)$. In fact the goal is to stabilize the system around a $p_d(t)$ which is as close as possible (in terms of magnitude and period) to a sinusoidal desired motion. Let us recall that the closed-loop system with fixed gains k_p, k_v yields at best a locally stable periodic trajectory which in general coexists with more complex dynamics. It is therefore an important matter to seek for a hybrid control strategy which eliminates this sort of behavior, and at the same time enlarges the basin of attraction of the desired motion as much as possible to counteract possible disturbances.

It is important to remark that there exists a basin of attraction around the fixed point of P corresponding to $p_d(t)$ since this corresponds to a hyperbolic fixed point of P. It is supposed that this region is not too small to be of practical interest.

The proposed control algorithm consists of 3 steps which can be enumerated,

1. Identify a periodic orbit (which is a fixed point for the impact Poincaré map of the closed-loop system with PD control), which is desirable in terms of system performance.

2. Direct the trajectory from an initial condition to a neighborhood around the desired fixed point representing a periodic orbit.

3. Switch to the k_p, k_v and ω values which locally stabilize the desired periodic orbit.

It is assumed that the switching is instantaneous and that the gains k_p, k_v and the frequency ω can be varied arbitrarily.

3.1 Constant-impulsive control

The approach to targeting the desired fixed point (x_1^*, y_1^*, y_2^*) consists in applying a constant control input at the impact time, and an impulsive input between two consecutive collisions to modify the velocity of the secondary mass. The algorithm is as follows: The primary mass starts in a constraint and a constant input λ_k is applied. The system evolves until an instant $t_i \in (t_k, t_{k+1})$ where an impulsive input p_k is applied to correct the velocity of the secondary mass. The procedure is applied many times as necessary to reach the desired target (m will be the number of times).

The equations describing the system between two consecutive impacts with the proposed control are:

$$x_1(t_{k+1}) = x_1(t_k) + y_1(t_k^+)\Delta_k \quad (8)$$

$$x_2(t_{k+1}) = x_2(t_k) + y_2(t_k^+)\Delta_k + \alpha_k p_k \Delta_k + \frac{\Lambda_k}{2}\Delta_k^2 \quad (9)$$

$$y_1(t_{k+1}^+) = m_{11} y_1(t_k^+) + m_{12} y_2(t_k^+) + m_{12}\Delta_k \Lambda_k + m_{12} p_k \quad (10)$$

$$y_2(t_{k+1}^+) = m_{21} y_1(t_k^+) + m_{22} y_2(t_k^+) + m_{22}\Delta_k \Lambda_k + m_{22} p_k \quad (11)$$

where $m_{11} = \frac{\mu - e}{1+\mu}$, $m_{12} = \frac{(1+e)}{1+\mu}$, $m_{21} = \frac{(1+e)\mu}{1+\mu}$, $m_{22} = \frac{1-\mu e}{1+\mu}$, $\Delta_k = t_{k+1} - t_k$, $\alpha_k = \frac{t_{k+1} - t_i}{\Delta_k}$ and $\Lambda_k = \frac{\lambda_k}{m_2}$.

One remarks from (8) that the control has no effect on the position of the primary mass because this is a free particle between collisions. In order to indirectly control the position of m_1 the flight time is used. The last impact is analyzed to determine the necessary conditions to attain the fixed point. To bring the position of the primary mass from $x_1(t_\kappa)$ to $x_1(t_{\kappa+1}) = x_1^*$, the elapsed time between the last two impacts must be given by:

$$\Delta_\kappa^* = \frac{x_1^* - x_1(t_\kappa)}{y_1(t_\kappa^+)} \quad (12)$$

Introducing (12) into (8)-(11) we obtain the system that provides the form of the controller on $(t_\kappa, t_{\kappa+1})$,

$$\begin{pmatrix} \frac{(\Delta^*)^2}{2} & \alpha_\kappa \Delta_\kappa^* \\ \Delta_\kappa^* & 1 \\ \Delta_\kappa^* & 1 \end{pmatrix} \begin{pmatrix} \Lambda_\kappa \\ p_\kappa \end{pmatrix} = \begin{pmatrix} x_2^* - x_2(t_\kappa) - y_2(t_\kappa^+)\Delta_\kappa^* \\ \frac{y_1^* - m_{11} y_1(t^+)}{m_{12}} - y_2(t_\kappa^+) \\ \frac{y_2^* - m_{21} y_1(t^+)}{m_{22}} - y_2(t_\kappa^+) \end{pmatrix} \quad (13)$$

It can be shown that the controller in (13) guarantees $\Delta_\kappa = \Delta_\kappa^*$ and $z(t_\kappa^+) = z^* = (x_1^* \; x_2^* \; y_1^* \; y_2^*)^T$ provided $z(t_\kappa^-)$ satisfies certain conditions as shown below.

In the set of equations (13) it is possible to observe that there exist more states that control inputs (this is an under-actuated system). However the last two equations are linearly dependent and the system has a solution if these equations are identical. We deduce that the next condition must be satisfied:

$$\frac{y_1^* - m_{11} y_1(t_\kappa^+)}{m_{12}} = \frac{y_2^* - m_{21} y_1(t_\kappa^+)}{m_{22}} \quad (14)$$

The equation (14) is verified only if $e = 0$ or $y_1(t_\kappa^+) = y_1^*$. This means that only for very particular initial conditions the control scheme brings the trajectory to the target in one collision (i.e. m=1). To cope with this problem, we define an intermediate state $(x_1^+, x_2^+, y_1^+, y_2^+)$ to be attained at t_κ from the initial state at $t_{\kappa-1}$. The idea is choose an intermediate state which has an existence region that contains the initial states $(x_1(t_{\kappa-1}), x_2(t_{\kappa-1}), y_1(t_{\kappa-1}^+), y_2(t_{\kappa-1}^+))$.

From the above we choose the intermediate state as follows:

$$\begin{cases} x_1(t_\kappa) = x_1^+ \\ x_2(t_\kappa) = x_2^+ \\ y_1(t_\kappa^+) = y_1^* \\ y_2(t_\kappa^+) = y_2^+ \end{cases} \quad (15)$$

To bring the trajectory from the initial state to the intermediate state the flight time is:

$$\Delta_{\kappa-1}^* = \frac{x_1^+ - x_1(t_{\kappa-1})}{y_1(t_{\kappa-1}^+)} \quad (16)$$

Introducing (16) into (13) and using only the two first equations we can obtain the values for $\Lambda_{\kappa-1}$ and $p_{\kappa-1}$ that bring the trajectory from the initial conditions to the intermediate state. The second equation is not considered and the velocity for m_2 is not controlled in this step. We obtain the control law on $(t_{\kappa-1}, t_\kappa)$:

$$\begin{pmatrix} \Lambda_{\kappa-1} \\ p_{\kappa-1} \end{pmatrix} = \begin{pmatrix} \frac{2(m_{12}\gamma_1 - \alpha_{-1} \; \Delta_{-1}^* \; \gamma_2)}{m_{12}(\Delta_{-1}^* \;)^2(1 - 2\alpha_{-1} \;)} \\ \frac{2 m_{12}\gamma_1 - \Delta_{-1}^* \; \gamma_2}{m_{12}\Delta_{-1}^* \; (2\alpha_{-1} \; -1)} \end{pmatrix} \quad (17)$$

where $\gamma_1 = x_2^+ - x_2(t_{\kappa-1}) - \Delta_{\kappa-1}^* y_2(t_{\kappa-1}^+)$ and $\gamma_2 = y_1^+ - m_{11} y_1(t_{\kappa-1}^+) - m_{12} y_2(t_{\kappa-1}^+)$.

In the second step, the system is considered on $(t_\kappa, t_{\kappa+1})$. From (8), $x_1(t_{\kappa+1}) = x_1^*$ implies:

$$\Delta_\kappa^* = \frac{x_1^* - x_1^+}{y_1^*} \quad (18)$$

To obtain the values for the control inputs, we solve the first and third equations of (13). It follows from the above choices of intermediate target that (11) and (10) are equivalent. Hence y_1^* and y_2^* are achieved simultaneously. One gets from (13):

$$\begin{pmatrix} \Lambda_\kappa \\ p_\kappa \end{pmatrix} = \begin{pmatrix} \frac{2(\alpha \ \Delta^* (y_2^* - m_{21} y_1^+ - m_{22} y_2^+) - m_{22}(x_2^* - x_2^+ - \Delta^* y_2^+))}{(\Delta^*)^2 m_{22}(2\alpha \ -1)} \\ \frac{2m_{22}(x_2^* - x_2^+ - \Delta_* \ y_2^+) - \Delta^* (y_2^* - m_{21} y_1^+ - m_{22} y_2^+)}{\Delta^* m_{22}(2\alpha \ -1)} \end{pmatrix} \tag{19}$$

The above developments are led assuming that there is no impact on the intervals $(t_{\kappa-1}, t_\kappa)$ and $(t_\kappa, t_{\kappa+1})$. Conditions on t_i, that is on α_k, are given next that guarantee such behavior.

3.1.1 Calculation of suitable impulse instants t_i (viability conditions)

The flight between the two constraints can be divided in two phases. The two phases will be considered independently. The equations governing the system during the first flight phase (t_k, t_i) are given by

$$\begin{aligned} y_1(t) &= y_1(t_k^+) && (20) \\ x_1(t) &= x_1(t_k) + y_1(t_k^+)(t - t_k) && (21) \\ y_2(t) &= y_2(t_k^+) + \Lambda_k (t - t_k) && (22) \\ x_2(t) &= x_2(t_k) + y_2(t_k^+)(t - t_k) + \frac{\Lambda_k}{2}(t - t_k)^2 && (23) \end{aligned}$$

In this phase the flight time between two collisions is given by:

$$\begin{aligned} \Delta_{ph1}^2 \frac{\Lambda_k}{2} + (y_2(t_k^+) - y_1(t_k^+))\Delta_{ph1} + (x_2(t_k) - x_1(t_k)) \quad (24) \\ = (x_2(t_p) - x_1(t_p)) \end{aligned}$$

where $\Delta_{ph1} = t_p - t_k$, t_p is the possible impact time (non desired) and $(x_2(t_p) - x_1(t_p))$ is the constraint of the non desired collision.

If $t_p > t_i$ (which will be satisfied if α and the inputs are suitably chosen, as shown later), the dynamical equations for the second flight phase (t_i, t_{k+1}) are

$$\begin{aligned} y_1(t) &= y_1(t_k^+) && (25) \\ x_1(t) &= x_1(t_k) + y_1(t_k^+)(t - t_k) && (26) \\ y_2(t) &= y_2(t_k^+) + \Lambda_k (t - t_k) + p_k && (27) \\ x_2(t) &= x_2(t_k) + y_2(t_k^+)(t - t_k) + p_k(t - t_i) && (28) \\ & \quad + \frac{\Lambda_k}{2}(t - t_k)^2 && \end{aligned}$$

In this phase the flight time between the constraints is given by:

$$\begin{aligned} \Delta_k^2 \frac{\Lambda_k}{2} + (\alpha p_k + y_2(t_k^+) - y_1(t_k^+))\Delta_k \quad (29) \\ + (x_2(t_k) - x_1(t_k)) = (x_2(t_{k+1}) - x_1(t_{k+1})) \end{aligned}$$

The equations (24) and (29) are quadratic. With the two constraints, each of them provides 4 different solutions depending on the impact constraints. This is a multiple valued problem and it is necessary to have a criterion to select the correct flight time.

flight time criteria The root represents a time interval between two consecutive collisions, it must be real and positive. A negative root or complex root are not feasible solutions and should be discarded.

If there exist multiple positive roots, there is a multiplicity problem that can be resolved only using the physical interpretation to the solutions. The solutions without physical meaning can be eliminated regarding the velocity value after the collision. There is only one possible sign depending upon the impact constraint. In this case the evaluation of the velocity gives a test which successfully eliminates the solutions without physical significance. It is important to consider that the multiplicity could be due to an inappropriate given initial state which does not represent a feasible state of the system.

It is noticeable that the form of the solution is given by the value of the inputs Λ and p. If one use the criteria of the flight time, it is possible to choose the impact constraint with an adequate selection of Λ and p. In this physical configuration there always exists a solution.

With this criteria, we can state the conditions for the non-existence of time t_p. They can be expressed as,

$$\Delta_{ph1} > \alpha_k \Delta_k \tag{30}$$

$$\Delta_k = \Delta_k^* \tag{31}$$

(30) expresses that the solution of (24) is larger than the interval (t_k, t_i). (31) means that the only solution of (29) (assuming that (30) is true) is the desired time flight time in (24) and (29) for the first and second phases respectively. The two equations displayed above are functions of α_k. We can simply find numerically the range of α_k satisfying these conditions. This procedure must be applied to the two steps of the algorithm.

Remark If there were no constraints, it would be possible to obtain the target in two steps from any initial condition, that is $m = 2$. However, the viability conditions imply a particular choice of the impulse instant t_i (i.e. of α_k), which reduces the size of the closed-loop basin of attraction. It is clear that if B_m denotes the basin of attraction for the control with m impacts, then $B_{m+1} \supseteq B_m$. But it remains to be proved that there exists M such that $B_M = R^4$, $M < +\infty$. Notice that the major control difficulty comes from the fact that each step one has to find a control input $U_k \in R^2$ such that (see (13))

$$M_k U_k = Z_k \qquad (32)$$

where $M_k \in R^{3\times 2}$, U_k and $Z_k \in R^3$ are straightforwardly defined from (13), and M_k depends nonlinearly on the state.

The figure 3 depicts a typical closed-loop trajectory with two steps. Figure 4 shows the set of starting states $(x_1^+, x_2^+, y_1^+, y_2^+)$ from which we can reach the desired target $(x_1^*, x_2^*, y_1^*, y_2^*) = (4.666, 3.666, -4.5333, -2.4)$ applying a one step constant-impulsive control, and with $y_1^+ = -y_1^*$. The depicted domain in the (x_2, y_2) plane therefore represents a section of the three dimensional basin of attraction corresponding to a one-step control. Notice that since $y_1^+ - y_2^+ > 0$ and $\Delta_{k+n} > 0$ it can be shown that the values of y_2 and x_2 belong to the domain $x_2^+ < x_1^* + L$, $y_2^+ < -y_1^*$.

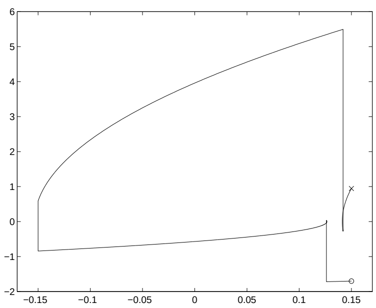

Figure 3: Constant-impulsive control (phase plane ($x = x_1 - x_2$, $y = y_1 - y_2$)).

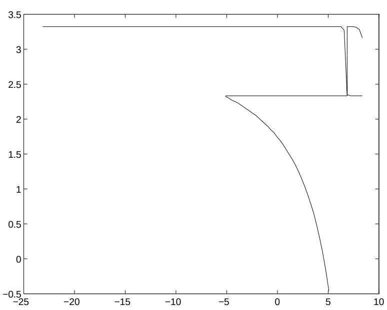

Figure 4: One step basin of attraction for the constant-impulsive control (section $y_1^+ = -y_1^*$)

4 Conclusions and future works

In this paper we have analyzed symmetric periodic motion of a simple mechanical system with dynamic backlash. The study of this problem including all its dynamic and non-linear features has been rarely treated in the control literature despite a lot of interest from the mechanics and non-linear dynamics community.

We have proposed a constant-impulsive hybrid controller that enlarges the basin of attraction of periodic orbits locally stable created by a PD controller.

In order to validate the practical applicability of the proposed control scheme we need to perform a robustness analysis of the closed-loop system. The main physical parameters which can affect the performance of the system are the coefficient of restitution, the mass ratio and the clearance length. We assume that the position and velocity of both the impacting bodies are available at all times which may not be completely practical. Perhaps also of limited practical value is the use of ideal impulses. We continue to study the use of non-impulsive control as well as the robustness issues.

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